Exercise 55

Let P and Q be polynomials. Find

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

Solution

Recall that the degree of a polynomial is the highest power of x.

Part (a)

If P is less than the degree of Q, then the limit will be, for example,

$$\begin{aligned}
\lim_{x\to\infty}\frac{P(x)}{Q(x)}=\lim_{x\to\infty}\frac{a+bx+cx^2+dx^3}{a+bx+cx^2+dx^3+ex^4+fx^5}&=\lim_{x\to\infty}\frac{a+bx+cx^2+dx^3}{a+bx+cx^2+dx^3+ex^4+fx^5}\cdot\frac{\frac{1}{x^5}}{\frac{1}{x^5}}\\
&=\lim_{x\to\infty}\frac{\frac{a}{x^5}+\frac{b}{x^4}+\frac{c}{x^3}+\frac{d}{x^2}}{\frac{a}{x^5}+\frac{b}{x^4}+\frac{c}{x^3}+\frac{d}{x^2}+\frac{e}{x}+f}\\
&=\frac{0+0+0+0}{0+0+0+0+f}\\
&=\frac{0}{f}\\
&=0.
\end{aligned}$$

Part (b)

If P is greater than the degree of Q, then the limit will be, for example,

$$\begin{aligned}
\lim_{x\to\infty}\frac{P(x)}{Q(x)}=\lim_{x\to\infty}\frac{a+bx+cx^2+dx^3+ex^4+fx^5}{a+bx+cx^2+dx^3}&=\lim_{x\to\infty}\frac{a+bx+cx^2+dx^3+ex^4+fx^5}{a+bx+cx^2+dx^3}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\\
&=\lim_{x\to\infty}\frac{\frac{a}{x^3}+\frac{b}{x^2}+\frac{c}{x}+d+ex+fx^2}{\frac{a}{x^3}+\frac{b}{x^2}+\frac{c}{x}+d}\\
&=\frac{0+0+0+d+\infty+\infty}{0+0+0+d}\\
&=\frac{\infty}{d}\\
&=\infty.
\end{aligned}$$